On the fine structure of turbulent flows

By R. BETCHOV
Department of Aeronautics, The Johns Hopkins University

(*Received* 18 *June* 1957)

Summary

The turbulence produced by a multiplicity of small air jets has been investigated and comparisons are made with other turbulent flows. The eddy Reynolds number is large ($R_\lambda = 250$).

The energy spectrum was measured, as well as the skewness and kurtosis of $\partial u/\partial x$. The effect of the finite length of the hot wire is considered and the corrected results indicate that the spectrum follows the law of Kolmogoroff in an intermediate range and appears to fall off with the (-6)-power of the frequency in the viscous range. This range is limited at the upper end of the frequency range by electrical noise. Special precautions reduced this noise to the level of thermal agitation in the hot wire.

The ultimate limit to spectral analysis is imposed by the molecular agitation of the gas. This limit is evaluated and compared with the spectrum of turbulence. It appears that the spectrum of $\partial^3 u/\partial x^3$ merges with the spectrum of molecular agitation without a distinctive separation.

1. Introduction

The fine structure of turbulent flows has been the object of numerous investigations but complete understanding has not yet been reached. In this paper, we describe certain recent results that may contribute to the clarification of the situation.

The theory of turbulence becomes greatly simplified if one assumes incompressibility, homogeneity and isotropy of the velocity field. Under such conditions, one can define a one-dimensional spectrum $F(k)$ which is a function of a wave-number k. Let us define k_E as the wave-number such that half the kinetic energy of the motion is contributed by the region $k > k_E$ and let us define k_ν as the wave-number such that half of the viscous energy dissipation occurs in the region $k > k_\nu$.

Two important theoretical predictions about $F(k)$ are as follows (Batchelor 1953, ch. 6). According to Kolmogoroff, the spectrum should fall as $k^{-5/3}$ in some subrange of wave-numbers in the range $k_E < k < k_\nu$, provided that $k_E \ll k_\nu$. According to Heisenberg, the spectrum should fall as k^{-7} in the region $k > k_\nu$. These two statements can be formulated in a single expression, and the theory shows that $F(k)$ should cross a limit given by $c\nu^2 k$ in the vicinity of $k = k_\nu$, where c is a universal constant and ν the kinematic viscosity. Physically, this means that spectral components such that $F < c\nu^2 k$ are strongly influenced by viscosity.

The theory of Heisenberg implies that the third derivative of the velocity does not exist, or, more exactly, that the mean-square of this quantity, given by the integral of $k^6 F(k)$, does not converge.

From Kolmogoroff's theory it can also be deduced that the parameter S (skewness of $\partial u/\partial x$) should be a constant, provided $k_E \ll k_\nu$.

The experimental studies of the fine structure of turbulence have been carried out in a single gas (air), virtually at a single temperature and pressure. The velocity fluctuations are sensed by hot-wire anemometers and the effect of the finite length of the wire is often disregarded. The hot-wire anemometer gives an electric signal proportional to the velocity fluctuation plus a certain amount of electronic noise, so that some kind of filter is desirable to reject unnecessary noise.

In general, the turbulent fluid passes by the hot-wire with a mean velocity U which is much larger than the velocity fluctuation and, with the hypothesis made by Taylor, the time derivative of the signal is taken as a fair approximation to the space derivative. This allows the measurement of quantities such as:

$$u' = \langle u^2 \rangle^{1/2}, \qquad \lambda = u' \langle (\partial u/\partial x)^2 \rangle^{-1/2},$$

where $\langle \; \rangle$ denotes a time average, u is the velocity fluctuation in the direction of the x-axis and λ is the dissipation length parameter. If $k_E \ll k_\nu$, it can be shown that $\lambda^{-3} \doteqdot k_E k_\nu^2$, and the eddy Reynolds number

$$R_\lambda = u'\lambda/\nu$$

becomes proportional to $(k_\nu/k_E)^{2/3}$. The verification of the theoretical laws thus requires a large value of R_λ, say $R_\lambda \geqslant 100$.

Grid flows

A large variety of investigations have been carried out in the turbulent wake of a plane grid obstructing a parallel flow. The requirements of homogeneity and isotropy are reasonably well satisfied here and for low velocities incompressibility is a reasonable approximation. R_λ usually varies from 20 to 100. In practice, it becomes difficult to observe the fine details of the motion when $R_\lambda > 60$. This is due to the fact that $u'/U \leqslant 0{\cdot}02$, resulting in a weak electric signal from the hot-wire. At some frequency ω_N the spectrum of the signal merges into the noise of the instruments, consequently the smallest observable details of the motion are comparable to the length $l_N = U/2\pi\omega_N$. With a given grid size and a given hot-wire, R_λ increases as $U^{1/2}$ whereas l_N increases at least as U. Therefore, as R_λ increases, the resolution of the fine structure deteriorates. To avoid this effect, one should use lower viscosities or larger wind tunnels. Because of these limitations, it was not possible either to really support or to disprove the theoretical spectral laws with the work on grid flow.

It has been found that S varies very little and that quantities such as $\langle (\partial u/\partial x)^2 \rangle$ and $\langle (\partial^2 u/\partial x^2)^2 \rangle$ are related to each other in the way required by the Navier–Stokes equations. Measurements of the quantity $\langle (\partial^3 u/\partial x^3)^2 \rangle$ have been reported by Batchelor & Townsend (1949), Townsend (1951), and Stewart & Townsend (1951); Liepmann, Laufer & Liepmann (1951) carried out similar investigations of the third derivative but report more cautiously that "it appears to be finite".

Townsend (1951) proposed a simple model for the fine structure of turbulence based on stretching of vortex sheets or vortex lines. It leads to a spectrum falling as an exponential function of k^2 when k is large. Measurements in grid flows in the range $15 \leqslant R_\lambda \leqslant 30$ are in good agreement with the model, but discrepancies seem to appear as R_λ increases.

Shear flows

The structure of turbulent shear flows has been studied by Laufer (1950, 1953), Klebanoff & Diehl (1951), Corrsin & Uberoi (1951) and others. The departure from isotropy and homogeneity are serious but it affects principally the spectral components with wave-numbers near k_E. In a flow with $R_\lambda = 300$, Laufer (1950) found that the fine structure is nearly isotropic, that a portion of the spectrum falls as $k^{-5/3}$ and that the high frequency end of the spectrum falls as k^{-7}. (He obtained this without applying a correction for the finite length effect of the hot-wire.) This last result implies that $\langle(\partial^3 u/\partial x^3)^2\rangle$ is infinite or at any rate that it is determined by values of k too large to be measured.

In an attempt to clarify this situation, we designed a simple device to produce turbulence with large R_λ and small l_N. The object of this paper is to describe the procedure and to comment on the results.

2. Experimental installation

The 'Porcupine'

A plywood box $60 \times 60 \times 120$ cm has five perforated sides, as shown on figure 1. The sixth side is faired into a duct leading to a 20 h.p. blower.

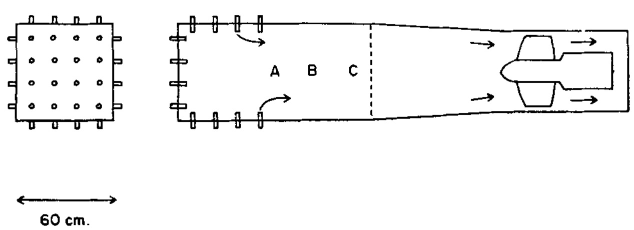

Figure 1. The 'Porcupine'. The mixing of 80 small jets produces a strong turbulence in the region marked *A, B, C*.

Each of the 80 openings is fitted with a round tube, 10 cm long and 2·5 cm inside diameter. In operation the air enters the box through these tubes, forming 80 jets which merge into a turbulent flow exhausted into the duct by the blower. The Reynolds number based on velocity and diameter of a single jet is about 35 000. The mean velocity of the flow in the parallel section is 450 cm/sec and the turbulence levels measured at the points A, B, C on the box axis are respectively 30%, 12% and 5%.

At point A, the turbulence is so strong that the hot-wire signal is not proportional to the velocity fluctuation, because of the intrinsic non-linearity of the instrument. At points B and C this distortion becomes less serious. The resolution length l_N is larger at B than at C by about 60%. and, as a compromise between the distortion and the loss of resolution, the author decided to perform all measurements at point B.

No systematic variations of U and u' over the plane normal to the mean flow were found, at distances up to 15 cm from the box axis. Both quantities varied by 5 to 15% between one run and another and it appears that the merging of the jets produces different flow patterns, as observed elsewhere by Corrsin (1944).

Hot-wire and preamplifiers

Pt wires (diameter 1·25 microns, length 1 mm) heated up to twice the cold resistance were used. Whenever it became essential to minimize the electric perturbations (60 cycles ripple or contact noise) each wire was heated with a separate 22 volts battery and a fixed wire-wound series resistor; all other metering devices were completely disconnected.

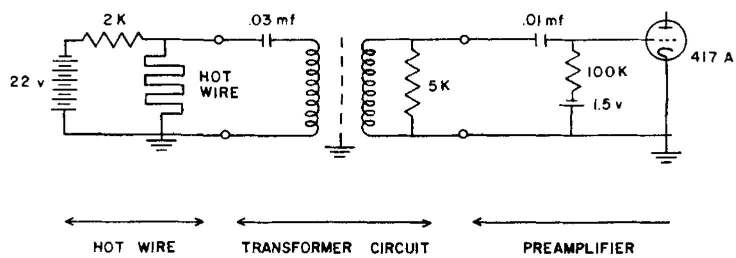

Figure 2. Input circuit for low electrical noise.

Two low noise preamplifiers were used, each with a Weston 417 A triode. The noise of each preamplifier is equivalent to that of a 600 ohms resistor applied to the input. With the amplifiers in parallel, a single channel is formed with a noise of 300 ohms. For measurements in the high frequency range, lower noise figures were achieved with the transformer input circuit of figure 2. The relatively small coupling condenser renders the output proportional to the time derivative of the signal, up to 20 kc/s. Above this frequency, the wire resistance and other impedances become significant and the overall response deteriorates. With a transformer ratio of 10 to 1, the input noise at 15 kc/s becomes equivalent to the thermal noise of a 50 ohms resistor at room temperature.

With an unheated wire of 60 ohms the total equivalent noise-generating resistance is 110 ohms. With the same wire heated to 120 ohms, that is, to approximately 600° K, we can expect a noise of about 300 ohms. This means that the amplitude of the noise should increase by about 260%, when the heating current is turned on. (This increase is noticeable on figure 5, at $k > 300$, in the range where the electronic noise dominates the turbulent signal.)

We verified that the battery did not make an additional contribution to the noise. For this purpose we replaced the hot-wire with a 200 ohms wire-wound resistor, and observed that the passage of the heating current did not affect the noise of the wire-wound resistor.

Thermal lag compensation

The thermal inertia of the hot-wire appreciably attenuates the signal above 600 c/s. With the transformer input of figure 2, the coupling condenser provides the desired compensation from 600 c/s to 20 kc/s. It would be inadvisable to modify this circuit and obtain a constant gain between 3 and 600 c/s. Indeed, the low frequency components of the signal would be so large that they would saturate the preamplifier.

With a capacitive coupling into the preamplifier, it was necessary to limit the response of the preamplifier to the range of 200 c/s to 20 kc/s, in order to avoid this saturation by low frequency components. When the transformer was not in use, the thermal compensation was provided by an operational amplifier with an appropriate feed-back loop. However, the presence of a large turbulent signal made it difficult to adjust the time constant of the circuit by the usual square wave method.

For measurements of u' or of the low frequency portion of the spectrum, the preamplifier was by-passed completely.

Differentiation and filtering

An operational amplifier was used in the manner of analogue computing. The response of each differentiating unit increased with ω up to 20 kc/s and decreased with ω^{-1} from 40 kc/s upward. This provided the appropriate filtering of the high frequency components of the electronic noise.

Wave analysers

Between 3 and 7500 c/s a General Radio wave analyser with a band-width proportional to the tuned frequency was used, and, from 100 c/s to 15 kc/s, a Hewlett Packard analyser built around a modulator and four amplifiers tuned for 20 kc/s. In order to avoid the formation of modulation images, it was necessary to insert ahead of this wave analyser a low-pass filter with cut-off at 15 kc/s. From 7 kc/s to 60 kc/s a Sierra analyser with constant band-width was used.

The output of each analyser was squared, averaged, and reduced to a constant band-width, and the responses were matched at 100 c/s and 10 kc/s. Good agreement was found in the two regions where analysers overlap.

When the spectrum of the signal falls very rapidly, an analyser may be unable to separate the components properly. Indeed a tuned amplifier driven with a frequency lower than the resonance frequency has a gain proportional to the frequency. With four tuned amplifiers and a signal whose spectrum falls as ω^{-8}, the output of the analyser is seriously in error. This difficulty can be avoided if the signal is differentiated once or twice before reaching the analyser, since each differentiation multiplies the spectrum by ω^2.

3. The energy spectrum (without correction for wire length)

With a wave analyser tuned on frequency ω the corresponding wave-number is $k = 2\pi\omega/U$ and the one-dimensional spectrum $F(k)$ is equal to the mean-square of the analyser output multiplied by a constant such that

$$\langle u^2 \rangle = \int_0^\infty F(k)\, dk.$$

In figure 3 we show $F(k)$ as a function of k, with linear scales. A check with a variable low-pass filter indicated that half of the mean-square of u is contributed by components below $k = 0{\cdot}1\ \mathrm{cm}^{-1}$ and this confirms the existence of a maximum of F. It also suggests an energy scale of the order of 10 cm, which is comparable with the size of the box. A check with two crossed wires indicated that $u'/v' = 1{\cdot}3$, where v' refers to a velocity component normal to the mean flow. This represents an appreciable deviation from isotropy.

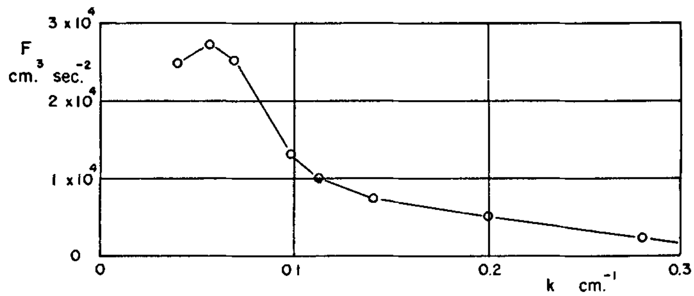

Figure 3. Energy spectrum *vs* wave-number with linear scales.

In figure 4 we show the spectrum with logarithmic scales. It follows the law of Kolmogoroff in the range $0{\cdot}15 \leqslant k \leqslant 15$. In the range $70 \leqslant k \leqslant 200$ it appears to follow the law of Heisenberg. For $k > 200$ the curve rises, but this is due to the electronic noise. These results have yet to be corrected for wire length effect; this will be done in the next section.

As previously discussed, it was suspected that the power law with which the spectrum falls off could be affected by the analyser itself. As a controlling experiment we measured the spectrum of $\partial^2 u/\partial t^2$, which is proportional to $k^4 F$. The results (see figure 5) show a corresponding portion of the curve falling as k^{-3}, in agreement with the measurements of $F(k)$ (for a wire of finite length).

Figure 5 shows the spectrum of the noise, measured with a cold wire. At high frequencies it remains slightly below the signal obtained with the heated wire, as expected.

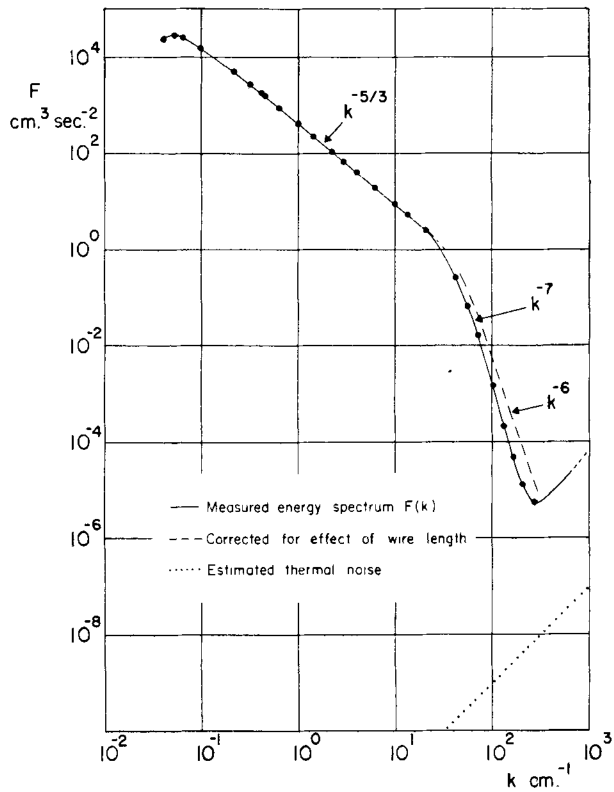

Figure 4. Energy spectrum *vs* wave-number with logarithmic scales.

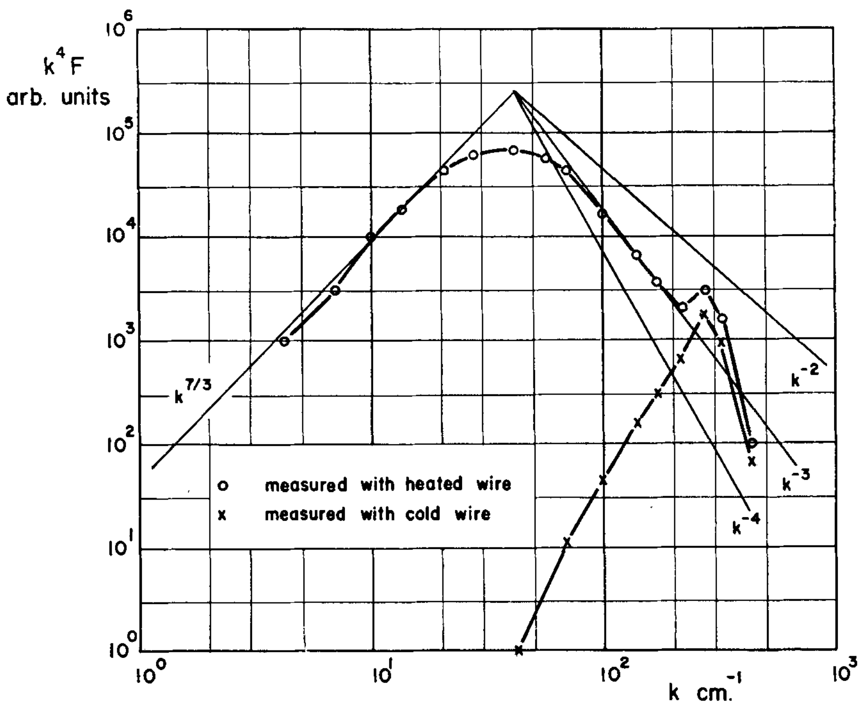

Figure 5. The spectrum of $\partial^2 u/\partial x^2$ (not corrected for wire-length effect).

In figure 6, the spectrum of figure 4 is compared with other results obtained in grid flows, in shear flows or in atmospheric turbulence. (It was necessary to convert the data published by several authors to the scales used here; this could not be done for the curves 7 and 8 and they have been given arbitrary translations.)

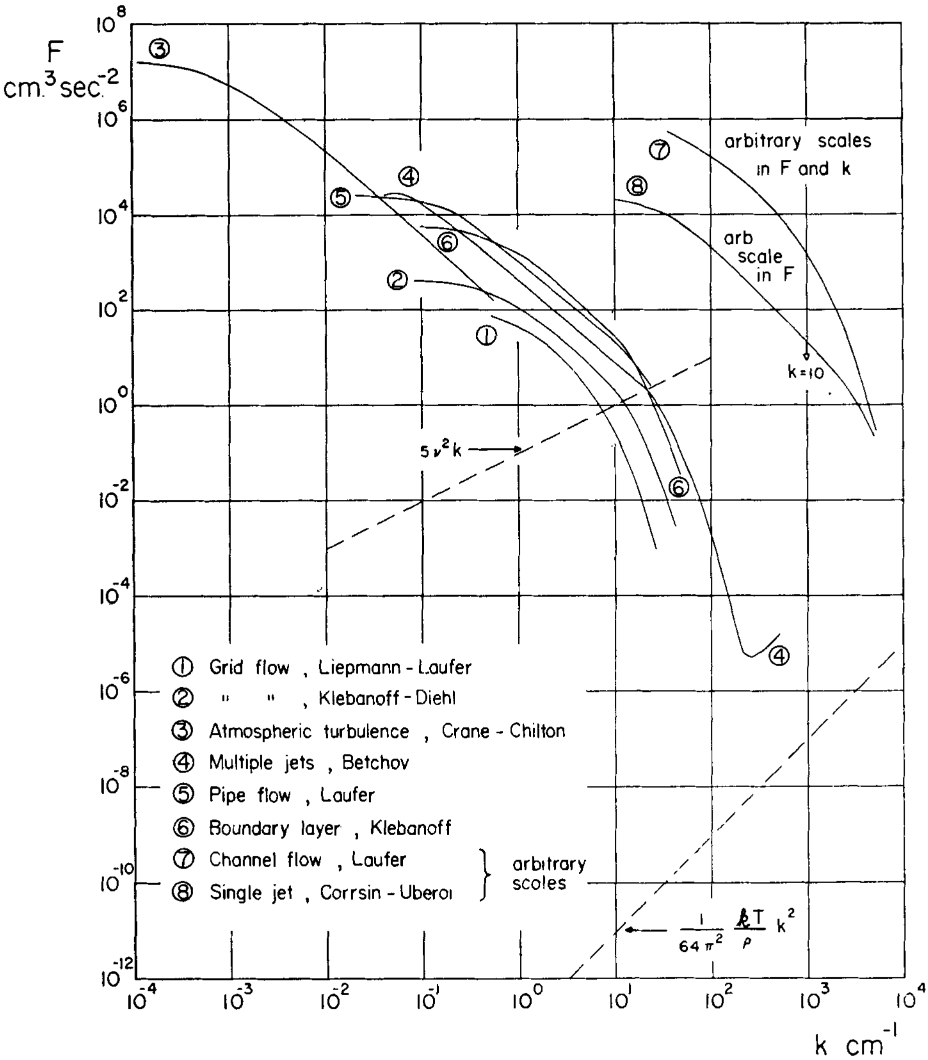

Figure 6. Comparison of several spectra.

We indicate by a dotted line the limit $c\nu^2k$ with the arbitrary choice $c = 5$. Below this line, the spectrum is controlled by viscosity (see § 1).

The grid flows nos. 1 and 2 (Liepmann, Laufer & Liepmann 1951; Klebanoff & Diehl 1951) appear to have energy scales too small for a well-developed range of Kolmogoroff's law. The shear flows nos. 5, 6, 7 (Laufer 1950, 1953; Klebanoff & Diehl 1951) and the author's turbulent flow no. 4 have such ranges. The curves 4, 5 and 6 seem to come together near $k = 20\ \text{cm}^{-1}$. As for atmospheric turbulence, no. 3 (Crane & Chilton 1956), the measurements have not been extended to the fine scale. (A pioneering contribution by Gödecke (1935) in the range $0{\cdot}1 \leqslant k \leqslant 10$ unfortunately does not yield the spectrum.) It would be interesting and not too difficult to study the fine structure of atmospheric turbulence; it might reveal a viscous cut-off somewhere beyond $k = 10$.

Beyond the viscous limit, all the spectra fall rapidly enough to ensure the existence of $\langle(\partial u/\partial x)^2\rangle$ and $\langle(\partial^2 u/\partial x^2)^2\rangle$. As for $\langle(\partial^3 u/\partial x^3)^2\rangle$, it appears to depend upon spectral components beyond the range of present measurements.

4. Correction for wire length

The finite length of the hot-wire introduces a particular type of error in the measurement of spectra. A complete theory of this effect has been given by Kovásznay & Uberoi (1953), with the assumption that all portions of the wire are equally sensitive. In the case of the present investigation, the 'cold ends' are about 0·05 mm wide each and the wire length is 1 mm, and, according to Betchov (1948), this means that a correction for finite length is necessary for $k > 20$.

In figure 4 a dotted line shows the approximate values of the corrected spectrum. Furthermore, the theory of the wire-length effect shows that, if the measured spectrum falls as k^{-7}, the true spectrum should fall as k^{-6}. Since this correction has not been applied to the results of figure 5, the apparent seventh power law must be interpreted as evidence that the true spectrum falls off as k^{-6}. This means that the law proposed by Heisenberg has no support from these experiments, and that the results of figure 5 suggest that the spectrum of the third derivative becomes flat for $k \geqslant 70$. There is little doubt that the correction for wire length applies also to Laufer's measurements (curve no. 7, figure 6).

As a subsidiary verification of the theory for wire-length effect, measurements $F(k)$ were made with two wires 1 and 6 mm long. The wave analyser was set on a particular frequency, and the two values of F were measured by simply switching from one wire to the other; this method minimizes the errors introduced by slow changes in the flow or the wire sensitivities. The same compensating network was used for both wires, since it was not possible to obtain an accurate adjustment of the compensating network in the presence of the large turbulence. The preamplifiers were operated with capacitive coupling (noise of 300 ohms), and therefore the measurements could not be extended beyond 15 kc/s. Figure 7 shows the measured values of η, where

$$\eta = \frac{[F(k)]_{l=6\,\text{mm}}}{[F(k)]^{l=1\,\text{mm}}}.$$

The theory of the wire-length effect predicts that η should start from 6 at low frequencies and begin to drop at $\omega = 250$ c/s, finally levelling off to $\sqrt{6}$ at high frequencies. The measurements support the theory.

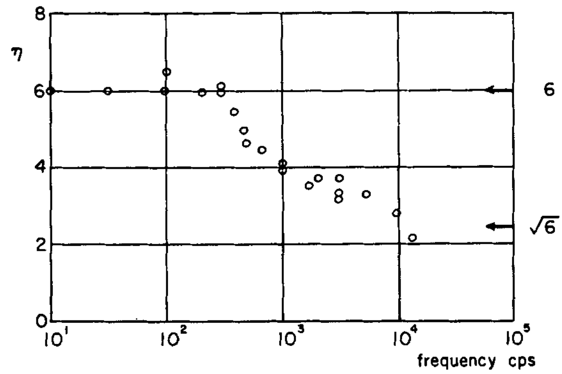

Figure 7. Effect of wire length on the measurement of a spectrum.

5. Skewness and kurtosis of $\partial u/\partial x$

Measurements of the mean-square of the square, cube and fourth powers of $\partial u/\partial x$ were made, using three chains of 10 double triodes, properly biased. Two chains were used for squaring and one for cubing, and the frequency responses were satisfactory. The largest errors were due to deviations from exact power law responses.

It was found that $S = -0{\cdot}45 \pm 0{\cdot}05$ and $\gamma = 4{\cdot}5 \pm 0{\cdot}5$; these results are not different from those obtained in grid flows with $30 < R_\lambda < 60$ or in shear flows (Betchov 1956).

The largest contribution to S originated from pulses of $\partial u/\partial t$ lasting approximately 0·5 millisecond. This corresponds to a length of about 2 mm and a correction for wire length may be necessary.

For the dissipation length parameter, the result was $\lambda = 0{\cdot}7$ cm which corresponds to a wave-number 1·5 cm^{-1} and to $R_\lambda = 250$. This result illustrates the fact that, at large values of R_λ, the length λ is intermediate between the energy scale k_E^{-1} (about 10 cm) and the scale of dissipation k_ν^{-1} (about 0·025 cm). As is now generally appreciated, the name 'microscale' which is sometimes given to λ is not appropriate.

6. Molecular agitation in a perfect gas

In these experiments, the spectrum could not be measured beyond the frequency at which the noise of the hot-wire becomes dominant. It can be shown theoretically that a hollow wire having the same outside diameter as the wire used here and heated to the same temperature would give a better signal-to-noise ratio. Thus, by removing metal from within the wire, one could in theory extend the measurements of $F(k)$, with no obvious limit until the atomic structure of the wire becomes significant.

This raises the question of other forms of noise and in particular of the noise produced by the gas itself. Somewhere, the spectrum of turbulence must merge with the spectrum of molecular agitation, and we shall now attempt to locate this point.

Let us consider a volume of gas of linear dimension L containing n molecules. The molecular velocities are of the order of $(k'T/m)^{1/2}$, where k' is Boltzmann's constant, T the absolute temperature and m the mass of a molecule. The velocity of the gas can be defined as that of the centre-of-mass of the n molecules and it will fluctuate with a mean-square velocity of the order of $k'T/mn$. Consider now two such elements of volume, with centres distance D apart. The correlation between the velocities of the two centres of mass will vary from 1 if $D = 0$ to zero if $D > 2L$. From the correlation we can compute the three-dimensional spectrum for $kL \ll 1$; in this way we find

$$E_{\text{noise}} = (3k'T/4\pi^2\rho)k^2,$$

where $\rho = nm/L^3$ is the gas density. This spectrum increases with k^2 and should reach a maximum near $kL = 1$. For the present purpose, we can assume that L is small (say $L = 1$ micron, or 50 mean-free-paths in air).

If the one-dimensional spectrum of turbulence F falls as k^{-6}, the three dimensional spectrum (Batchelor 1953) is given by $E_{\text{turb.}} = 48F$. The spectrum of turbulence can therefore be expected to cross that of molecular agitation (so that $E_{\text{turb.}} = E_{\text{noise}}$) at the wave-number where

$$F(k) = (k'T/64\pi^2\rho)k^2. \tag{1}$$

This expression should be regarded as a crude estimate. It is based on a simplified picture of molecular agitation which does not include fluctuations of pressure, density or temperature. It is also based on the application of the Taylor hypothesis to the finest components of the turbulence. This certainly ceases to be valid of $k > U/\nu$, since such eddies are dissipated before they are displaced by the mean flow. In the present case this occurs at $k > 3000\ \text{cm}^{-1}$.

The expression on the right-hand side of (1) is shown as dotted lines on figures 4 and 6 and it suggests that, unless $F(k)$ has an abrupt cut-off, the spectrum of turbulence disappears in molecular noise with a logarithmic slope of -6. This would leave quantities such as $\langle(\partial^3u/\partial x^3)^2\rangle$ without significance in the sense of fluid mechanics. Stating it a different way, it appears that the gap between turbulence and molecular agitation may not be as wide as is generally assumed.

This work was jointly sponsored by the Navy Bureau of Ordnance under contract Nord 15872, and Project SQUID, which is jointly supported by the Office of Naval Research, the Office of Scientific Research (Air Force), and the Office of Ordnance Research (Army) under contract N6ori-105 TO III.

References

BATCHELOR, G. K. 1953 *The Theory of Homogeneous Turbulence.* Cambridge University Press.

BATCHELOR, G. K. & TOWNSEND, A. A. 1949 The nature of turbulent motion at large wave-numbers, *Proc. Roy. Soc.* A, **199**, 238.

BETCHOV, R. 1948 L'influence de la conduction thermique sur les anemometres à fil chaud, *Proc. Kon. Akad. Wet.* **51**, 721.

BETCHOV, R. 1956 An inequality concerning the production of vorticity in isotropic turbulence, *J. Fluid Mech.* **1**, 497.

CRANE, H. L. & CHILTON, R. G. 1956 Measurements of atmospheric turbulence over a wide range of wave-length for one meteorological condition, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 3702.

CORRSIN, S. 1944 Investigation of the behavior of parallel two-dimensional air jets, *Nat. Adv. Comm. Aero., Wash., War Rep.* ACR 4H24.

CORRSIN, S. & UBEROI, M. S. 1951 Spectra and diffusion in a round turbulent jet, *Nat. Adv. Comm. Aero., Wash., Rep.* no. 1040.

GÖDECKE, K. 1935 Messungen der atmosphärischen Turbulenz in Bodennähe, *Ann. Hydr., Berl.,* **63**, 400.

KLEBANOFF, P. S. & DIEHL, Z. W. 1951 Some features of artificially fully developed turbulent boundary layers with zero pressure gradients, *Nat. Adv. Comm. Aero., Wash., Tech. note* no. 2475.

KOVÁSZNAY, L. S. G. & UBEROI, M. S. 1953 On mapping and measurements of random fields, *Quart. Appl. Math.* **10**, 375.

LAUFER, J. 1950 Investigation of turbulent flow in a two-dimensional channel, *Nat. Adv. Comm. Aero., Wash., Tech. note* no. 2123.

LAUFER, J. 1953 The structure of turbulence in fully developed pipe flow, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 2954.

LIEPMANN, H. W., LAUFER, J. & LIEPMANN, K. 1951 On the spectrum of isotropic turbulence, *Nat. Adv. Comm. Aero., Wash., Tech. Note* no. 2473.

STEWART, R. W. & TOWNSEND, A. A. 1951 Similarity and self-preservation in isotropic turbulence, *Phil. Trans.* A, **243**, 359.

TOWNSEND, A. A. 1951 On the fine-scale structure of turbulence, *Proc. Roy. Soc.* A, **208**, 534.